

Simple Calculation of Instanton Corrections in Massive $N=2$ $SU(3)$ SYM

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Abstract

We give an explicit derivation of the Picard-Fuchs equations for $N = 2$ supersymmetric $SU(3)$ Yang-Mills theory with $N_f < 6$ massive hypermultiplets in the fundamental representation. We determine the instanton corrections to the prepotential in the weak coupling region using the relation between $\text{Tr}\langle\phi^2\rangle$ and the prepotential. This method can be generalized straightforwardly to other gauge groups.

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1 Introduction

There has been much progress in duality in $N = 2$ supersymmetric field theory as well as in string theory, initialized by Seiberg and Witten [1]. The main idea to solve the effective theory by means of duality is to introduce a family of auxiliary curves. The moduli space of the curves coincides with the quantum moduli space of the gauge theory.

In $N = 2$ supersymmetric gauge theory with gauge group G the low energy Wilsonian effective action can be described by a single holomorphic prepotential \mathcal{F} which in $N = 1$ language is a function of rank G chiral multiplets A_i with scalar components a_i . The metric on the moduli space is $ds^2 = \text{Im} da_{Di} d\bar{a}_i$ where $a_{Di} = \frac{\partial \mathcal{F}}{\partial a_i}$ are the magnetic duals of a_i . a_D and a can be calculated as period integrals of a meromorphic differential on an auxiliary complex algebraic curve. Once we know $a_D(a)$, the prepotential can be obtained by integration and is in the semiclassical region of the form [2]:

$$\begin{aligned}\mathcal{F}_{\text{class}} &= \frac{\tau_{\text{class}}}{2} \sum_{\alpha \in \Delta_+} \langle \alpha, a \rangle^2 \\ \mathcal{F}_{1\text{-loop}} &= \frac{i}{4\pi} \left\{ \sum_{\alpha \in \Delta_+} \langle \alpha, a \rangle^2 \ln \frac{\langle \alpha, a \rangle^2}{\Lambda^2} - \frac{1}{2} \sum_{i=1}^{N_f} \sum_w (\langle w, a \rangle + m_i)^2 \ln \frac{(\langle w, a \rangle + m_i)^2}{\Lambda^2} \right\} \\ \mathcal{F}_{\text{inst}} &= \sum_{n=1}^{\infty} \mathcal{F}_n(a) \Lambda^{(2N_c - N_f)n}\end{aligned}\tag{1}$$

The sums are over the positive roots Δ_+ of G and the weights of the representation of the N_f hypermultiplets, respectively. In the case of $SU(3)$ the positive roots in Dynkin basis are $(2, -1)$, $(-1, 2)$ and $(-1, -1)$, and the weights of the fundamental representation $\underline{3}$ are $(1, 0)$, $(0, -1)$ and $(-1, 1)$. Generalizations of this scheme to various gauge groups without [3, 4, 5] and with matter [6, 7, 8, 9, 10, 11, 12, 13] have been given.

Although the periods and the prepotential can be obtained by explicit integration to compute the one and two instanton corrections [2, 14], for the $SU(2)$ case see also [15, 16], the approach via the Picard-Fuchs (PF) equations for the periods leads also to the instanton corrections and can easily be pushed to arbitrary orders. PF-operators for the case of A_r with $r \leq 3$ without matter have been derived in [5], a closed form for the Lie groups A_r , B_r , C_r and D_r has been given in [17]. The cases of massless matter for the groups with $r \leq 3$ were done in [18, 19, 20, 21]. $SU(2)$ with massive matter was investigated in [22].

In this paper we first construct a set of PF-operators for $SU(3)$ with $N_f < 6$ massive hypermultiplets in the fundamental representation. From their power series solutions we find their periods and calculate the instanton corrections to the prepotential in the weak coupling region $u \rightarrow \infty$ using the method outlined in [20], which relies on the relation between the second Casimir u and the prepotential [23, 24, 25]. This method should be easily applicable to other Lie groups as well.²

² The Reduce procedures which were used to derive the results in this paper may be obtained by e-mail request from the authors.

2 Picard-Fuchs Operators for Massive Matter

We consider $N = 2$ supersymmetric YM theory with gauge group $SU(3)$ and $N_f < 6$ massive hypermultiplets in the fundamental representation. The appropriate hyperelliptic curves, which define a Riemann surface of genus two are [11]:

$$y^2 = W^2(x; u, v) - F(x; m, \Lambda) \quad (2)$$

where $W = (x^3 - ux - v)$ is the A_2 simple singularity, with $u = \text{Tr}\langle\phi^2\rangle$ and $v = \text{Tr}\langle\phi^3\rangle$, where $\phi = \sum a_i H_i$, with H_i the generators of the Cartan subalgebra. The mass dependent part of the curve is:

$$F(x; m, \Lambda) = \Lambda^{2N_c - N_f} \left(x^{N_f} + \sum_{i=1}^{N_f} t_i x^{N_f - i} \right) \quad (3)$$

with $t_i = \sum_{j_1 < \dots < j_i} m_{j_1} \dots m_{j_i}$. For $N_f < 5$ the m_i coincide with the hypermultiplet masses M_i while in the case $N_f = 5$ there is a constant shift $m_i = M_i - \frac{\Lambda}{12}$. Λ is meant to be Λ_{N_f} .³

The meromorphic differential associated to the hyperelliptic curves is [9, 12]:

$$\lambda = \frac{x}{2i\pi y} \left(W' - \frac{1}{2} \frac{WF'}{F} \right) dx \quad (4)$$

and the periods are integrals over λ : $a_i = \int_{\alpha_i} \lambda$ and $a_{Di} = \int_{\beta_i} \lambda$ where α_i and β_i are a symplectic basis of homology 1-cycles on the curves (2), i.e. $\alpha_i \cap \beta_j = -\beta_j \cap \alpha_i = \delta_{ij}$ and $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = 0$. First derivatives of λ with respect to u and v yield abelian differentials of the first kind:

$$\begin{aligned} \partial_v \lambda &= \frac{1}{2\pi i y} dx \\ \partial_u \lambda &= \frac{x}{2\pi i y} dx \end{aligned} \quad (5)$$

For the integrals over the abelian differentials of the first kind (5) we can find PF-operators by considering the partial derivatives $\partial_u, \partial_v, \partial_{uu}^2, \partial_{uv}^2$ and ∂_{vv}^2 of the differentials, producing expressions of the form $\frac{\phi(x)}{y^n}$ where $\phi(x)$ is some polynomial in x . Explicitly we get:

$$\begin{aligned} \partial_u \left(\frac{1}{y} \right) &= \frac{xW}{y^3} & \partial_v \left(\frac{1}{y} \right) &= \frac{W}{y^3} \\ \partial_{uu}^2 \left(\frac{1}{y} \right) &= \frac{2x^2}{y^3} + \frac{3x^2 F}{y^5} & \partial_{uv}^2 \left(\frac{1}{y} \right) &= \frac{2x}{y^3} + \frac{3xF}{y^5} & \partial_{vv}^2 \left(\frac{1}{y} \right) &= \frac{2}{y^3} + \frac{3F}{y^5} \end{aligned} \quad (6)$$

The corresponding expressions for the partial derivatives applied to $\frac{x}{y}$ differ by one additional power of x in the numerator.

³ The curves derived in [8, 13] coincide with the ones taken here for $N_f = 1$ and $N_f = 2$ while for $N_f \geq 3$ they differ in the expression for $W(x; u, v)$. The superscript K indicates the convention taken in [13]: $W^K = W + (\Lambda^{2N_c - N_f}/4) \sum_{i=0}^{N_f - N_c} t_i x^{N_f - N_c - i}$, with $t_0 = 1$. These curves are related to ours by an appropriate transformation of u, v, t_i and Λ . In particular the shift in u for $N_f = 4$ to $u^K = u + \Lambda^2/4$ and for $N_f = 5$ to $u^K = u + \Lambda t_1/4$ will turn out to be significant for the comparison of the instanton corrections.

In [20] we reduced these expressions with the help of two identities to a basis of differentials:

$$\frac{dx}{y}, \dots, \frac{x^4 dx}{y} \quad (7)$$

This basis consists of the abelian differentials of first kind $\frac{dx}{y}$, $\frac{x dx}{y}$, of second kind $\frac{x^3 dx}{y}$, $\frac{x^4 dx}{y}$ and of third kind $\frac{x^2 dx}{y}$. The identities which allow the reduction to the basis above are valid up to total derivatives [5, 20]. To reduce the degree of x in the numerator we use:

$$\frac{x^k}{y^l} = \frac{x^{k-6}}{y^l} \frac{((l-2)x\varphi - 2(k-5)\psi)}{2-6l+2k} \quad (8)$$

where φ and ψ are to be taken from $y^2 = x^6 + \psi(x)$ and $(y^2)' = 6x^5 + \varphi(x)$. The formula (8) only works for $k \neq 3l-1$. Notice that this condition means no restriction in the application to $\frac{x^k}{y}$, with $k > 4$, always allowing to reduce such monomials to the basis (7). The reduction of powers of y in the denominator is done by

$$\frac{\phi(x)}{y^l} = \frac{1}{\Delta y^{l-2}} \left\{ a\phi + \frac{2}{l-2} (b\phi)' \right\} \quad (9)$$

which requires the decomposition of the discriminant of the curve Δ_{N_f} into $\Delta_{N_f} = a(x)y^2 + b(x)(y^2)'$, resulting in huge expressions in the cases of massive matter.

Since the expressions (6) contain $1/y^3$ and $1/y^5$ one might as well take into consideration a different set of basic differentials based on x^k/y^3 or x^k/y^5 . The power of $1/y$ can easily be increased by multiplication with the curve:

$$\frac{\phi(x)}{y^l} = \frac{(W^2 - F)\phi}{y^{l+2}} \quad (10)$$

followed by repeated application of (8). Doing this we arrive at differentials of the form:

$$\frac{dx}{y^3}, \dots, \frac{x^4 dx}{y^3}, \frac{x^8 dx}{y^3} \quad (11)$$

or differentials depending on $1/y^5$

$$\frac{dx}{y^5}, \dots, \frac{x^4 dx}{y^5}, \frac{x^{14} dx}{y^5} \quad (12)$$

These sets represent no longer a basis of differentials but form an overcomplete set. The dependence among the differentials in (11) or (12) corresponds just to the case where equation (8) fails to work. Actually equation (8) allows to find the total differential which gives rise to the relation by multiplying (8) on both sides with $(2-6l+2k)$ and considering the case where the left hand side vanishes. This identifies

$$\frac{x^3\varphi(x) - 6x^2\psi(x)}{y^3} \quad \text{and} \quad \frac{x^9\varphi(x) - 6x^8\psi(x)}{y^5} \quad (13)$$

as total differentials, to which the same equation (8) can be applied to reduce all powers of x to the ones contained in (11) or (12). We will denote these total differentials by r_{N_f} . The actual expressions for r_{N_f} depend on N_f . The most convenient choice of a basic set of differentials for deriving PF-operators turns out to be the set (12) since this avoids the use of the discriminant and its decomposition.

In general, PF-operators constitute a system of partial differential operators of second order for abelian differentials of the first kind. If we consider first the differential $\partial_v \lambda$ the PF equations appear as

$$\begin{aligned} 0 = \mathcal{L}_{(i)} \int \partial_v \lambda &= \left(c_1^{(i)} \partial_{uu}^2 + c_2^{(i)} \partial_{uv}^2 + c_3^{(i)} \partial_{vv}^2 + c_4^{(i)} \partial_u + c_5^{(i)} \partial_v + c_6^{(i)} \right) \int \partial_v \lambda \\ &= \int \left\{ (c_1^{(i)} \partial_{uu}^2 + c_2^{(i)} \partial_{uv}^2 + c_3^{(i)} \partial_{vv}^2 + c_4^{(i)} \partial_u + c_5^{(i)} \partial_v + c_6^{(i)}) \partial_v \lambda + c_x^{(i)} r_{N_f} \right\} \end{aligned} \quad (14)$$

with $i = 1, 2$. The coefficients $c_j^{(i)}$ are polynomial functions in u and v and the normalization might be chosen as $c_3^{(1)} = 0$ and $c_1^{(2)} = 0$. Applying the partial derivatives to $\partial_v \lambda$ and reducing the resulting expressions to the differentials (12) by the procedure described above, i.e. by the rules (8) and (10), we find after inclusion of the total derivative r_{N_f} two vanishing nontrivial linear combinations

$$c_1^{(i)} \overline{\partial_{uu}^2 \partial_v \lambda} + c_2^{(i)} \overline{\partial_{uv}^2 \partial_v \lambda} + c_3^{(i)} \overline{\partial_{vv}^2 \partial_v \lambda} + c_4^{(i)} \overline{\partial_u \partial_v \lambda} + c_5^{(i)} \overline{\partial_v \partial_v \lambda} + c_6^{(i)} \overline{\partial_v \lambda} + c_x^{(i)} r_{N_f} = 0 \quad (15)$$

where overlining denotes the reduction to the basis (12), yielding the two PF-operators. Similarly we determine PF-operators $\hat{\mathcal{L}}_{(i)}$ for the differential $\partial_u \lambda$ with $\hat{\mathcal{L}}_{(i)} \int \partial_u \lambda = 0$.

In contrast to the massless case the PF operators for the periods of the meromorphic differential (4) are third order differential operators. More precisely, the section $\Pi = (\vec{a}_D, \vec{a})^T$ does not transform irreducibly under monodromy and consequently there is no $\tilde{\mathcal{L}}$ of second order with $\partial_v \tilde{\mathcal{L}} \Pi = 0$.

In the appendix we give as an example the complete set of PF-operators for $N_f = 1$. With increasing N_f the PF-operators become much larger, but factorize in the case of equal masses through powers of $(m^3 - mu + v)$ resulting in expressions which are of similar size as the ones given.

3 Power Series Solutions and Instanton Corrections

In the weak coupling region of the moduli space the PF-equations $\mathcal{L}_{(i)} \omega(u, v)$ and $\hat{\mathcal{L}}_{(i)} \omega(u, v)$ are known to have two pure power series solutions and two solutions including logarithms each. To calculate the instanton corrections we apply the method we derived in [20] which requires only the power series solutions. The instanton corrections in the semiclassical patches for $u \rightarrow \infty$ and $v \rightarrow \infty$ coincide. For our method we will start from the power series solutions in the semiclassical region $u \rightarrow \infty$, which are found by an ansatz of the form

$$\omega(u, v) = u^{-k} v^l \sum_{m,n=0}^{\infty} c_{mn}(\Lambda, t_i) u^{-m} v^n \quad (16)$$

where the rational indices (k, l) have to be determined as part of the solution. They turn out to be $(\frac{5}{2}, 0)$ $((\frac{3}{2}, 0)$ for $N_f = 5)$ and $(1, 0)$ for $\mathcal{L}_{(i)}$ and $(\frac{1}{2}, 0)$, $(2, 0)$ for $\hat{\mathcal{L}}_{(i)}$.

At weak coupling we denote the two power series solutions of the PF-operators $\mathcal{L}_{(i)}$ and $\hat{\mathcal{L}}_{(i)}$ by ω_1, ω_2 and ω_3, ω_4 , respectively. The derivatives of the periods a_i with respect to u and v are linear combinations of the power series solutions of the PF-operators. The periods a_{Di} also depend on the logarithmic solutions. Let us consider a_1 first. We have:

$$\begin{aligned}\partial_v a_1 &= \rho_1 \omega_1 + \rho_2 \omega_2 \\ \partial_u a_1 &= \rho_3 \omega_3 + \rho_4 \omega_4\end{aligned}\tag{17}$$

where ρ_i are some constants which will be determined in the following. Equations (17) give rise to an integrability condition [5] by which we can eliminate two of the constants, e.g. ρ_3 and ρ_4 . Integrating the system (17) we can determine $a_1(u, v)$ up to a constant. The resulting expression for a_1 still depends on ρ_1 and ρ_2 . a_2 is analogous but with different values for the constants ρ_1 and ρ_2 .

To proceed we introduce the expressions u_0, v_0 and $\Delta_0 = 4u_0^3 - 27v_0^2$:

$$\begin{aligned}u_0 &= a_1^2 + a_2^2 - a_1 a_2 \\ v_0 &= a_1 a_2 (a_1 - a_2) \\ \Delta_0 &= \prod_{\alpha \in \Delta_+} \langle \alpha, a \rangle^2\end{aligned}\tag{18}$$

where u_0 reproduces u and v_0 reproduces v up to higher order corrections in Λ . In the limit $\Lambda \rightarrow 0$ these are just the equations for the classical Casimirs and the classical discriminant.

Now it remains to fix the coefficients ρ_1 and ρ_2 . Inverting u_0 and v_0 in the semiclassical region around $u_0 \rightarrow \infty$ we find that to leading order⁴

$$a_1 = \sqrt{u_0} + \frac{1}{2} \frac{v_0}{u_0} + \dots \quad \text{and} \quad a_2 = \sqrt{u_0} - \frac{1}{2} \frac{v_0}{u_0} + \dots\tag{19}$$

In the quantized theory the coefficients ρ_1 and ρ_2 can thus be fixed by adjusting the coefficients of \sqrt{u} and $\frac{v}{u}$ in a_i to the values 1 and $\pm \frac{1}{2}$, respectively. Indeed we find after integrating (17) that the a_i start with $\rho_1 \sqrt{u} + \rho_2 \frac{v}{u} + \dots$. This completes the determination of the periods in terms of the power series solutions of the PF-operators.

We now use the relation between u and the prepotential \mathcal{F} derived in [23, 24, 25] to calculate the instanton corrections for the theories with $N_f < 6$ massive hypermultiplets. Since the prepotential \mathcal{F} is a homogeneous function of weight 2 in m_i, a_i and Λ it satisfies the Euler equation

$$2\mathcal{F} = \Lambda \frac{\partial \mathcal{F}}{\partial \Lambda} + \sum_{i=1,2} a_i \frac{\partial \mathcal{F}}{\partial a_i} + \sum_{i=1}^{N_f} m_i \frac{\partial \mathcal{F}}{\partial m_i}\tag{20}$$

⁴The normalization taken here differs from the one in [20] due to a factor $\frac{1}{2}$ in a_i . The relation between the conventions of that paper marked by E and the ones taken here are: $\Delta_0 = \frac{\Delta_0^E}{4^3}$ and $v_0 = -\frac{v_0^E}{8}$ and $u_0 = \frac{u_0^E}{4}$.

It can be shown that

$$\partial_u \left(\Lambda \frac{\partial \mathcal{F}}{\partial \Lambda} \right) = \frac{2N_c - N_f}{2\pi i} \quad (21)$$

holds even in the massive case. Integrating this formula gives u up to a function $c(v, m, \Lambda)$. On the other hand, taking the derivative of (1) with respect to Λ leads to

$$\Lambda \frac{\partial \mathcal{F}}{\partial \Lambda} = \frac{1}{2\pi i} (2N_c - N_f) u_0 - \frac{N_c}{4\pi i} \sum_{i=1}^{N_f} m_i^2 + (2N_c - N_f) \sum_{n=1}^{\infty} \mathcal{F}_n n \Lambda^{(2N_c - N_f)n} \quad (22)$$

Comparing the integrated equation (21) with the equation above we get

$$u(a) = u_0 + 2i\pi \sum_{n=1}^{\infty} \mathcal{F}_n(a) n \Lambda^{(2N_c - N_f)n} + \Lambda^{2N_c - N_f} \tilde{c}(v, m, \Lambda) \quad (23)$$

where we have fixed the Λ independent part of $c(v, m, \Lambda)$ to $-\frac{N_c}{4i\pi} \sum_{i=1}^{N_f} m_i^2$ by using the fact that in the classical limit u and u_0 coincide. The remaining part of c is strongly restricted by R -charge considerations. Since u has charge 4 and only positive powers of m with charge 2 and v with charge 6 are supposed to appear, the only possible terms are Λ^2 and Λm . These can appear for $N_f = 4$ and $N_f = 5$, whereas in all other cases \tilde{c} must vanish.

The remaining freedom in the expression for u corresponds to a shift of \mathcal{F}_1 in both cases. It is just this shift by which one instanton results in the literature [2, 19, 20] differ. The results in [2, 19] were derived by explicit integration from curves given in [8, 13] which are different from each other and from the ones we use [11]. We find that our procedure reproduces the results starting from the corresponding curve if we set \tilde{c} to zero in each case. The shift in u is the same as the one discussed in footnote 3.

After determining the coefficients \mathcal{G}_n in

$$u - u_0(u, v; m, \Lambda) = \sum_{n=1}^{\infty} \mathcal{G}_n(a) \Lambda^{(2N_c - N_f)n} \quad (24)$$

the instanton corrections to the prepotential are obtained by comparing the two series.

Each individual correction is a finite expression (as opposed to an infinite series in u and v) if we express it in powers of u_0 , v_0 and Δ_0^{-1} . The remaining constant from integrating (17) is immediately fixed by demanding convergence and turns out to be different from zero only for $N_f = 5$.

For $N_f = 1$ we give the one, two and three instanton correction:

$$\begin{aligned} \mathcal{F}_1^{N_f=1} &= \frac{1}{2\pi i} \frac{3}{4\Delta_0} (-3v_0 + 2m_1 u_0) \\ \mathcal{F}_2^{N_f=1} &= \frac{1}{4\pi i} \left\{ \frac{3645v_0^2}{32\Delta_0^3} (-9v_0 m_1 + u_0^2 + 3u_0 m_1^2) + \frac{9}{32\Delta_0^2} (17m_1^2 u_0 - 111m_1 v_0 + u_0^2) \right\} \\ \mathcal{F}_3^{N_f=1} &= \frac{1}{6\pi i} \left\{ \frac{3}{64\Delta_0^3} (770m_1^3 u_0 - 11457m_1^2 v_0 + 266m_1 u_0^2 - 367u_0 v_0) \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{2187}{128\Delta_0^4} v_0^2 (442m_1^3 u_0 - 2961m_1^2 v_0 + 346m_1 u_0^2 - 239u_0 v_0) \\
& + \frac{14348907}{128\Delta_0^5} v_0^4 (2m_1^3 u_0 - 9m_1^2 v_0 + 2m_1 u_0^2 - u_0 v_0) \}
\end{aligned}$$

For $N_f = 2$ we get ($t_1 = m_1 + m_2$, $t_2 = m_1 m_2$):

$$\begin{aligned}
\mathcal{F}_1^{N_f=2} &= \frac{1}{2\pi i} \frac{1}{4\Delta_0} (2u_0^2 - 9v_0 t_1 + 6u_0 t_2) \\
\mathcal{F}_2^{N_f=2} &= \frac{1}{4\pi i} \left\{ \frac{5}{128\Delta_0} + \frac{9}{64\Delta_0^2} (2t_1^2 u_0^2 - 222t_1 t_2 v_0 - 46t_1 v_0 u_0 + 34t_2^2 u_0 + 28t_2 u_0^2 + 51v_0^2) \right. \\
& + \frac{3645v_0^2}{128\Delta_0^3} (4t_1^2 u_0^2 - 36t_1 t_2 v_0 - 12t_1 v_0 u_0 + 12t_2^2 u_0 + 8t_2 u_0^2 + 9v_0^2) \} \\
\mathcal{F}_3^{N_f=2} &= \frac{1}{6\pi i} \left\{ \frac{3}{256\Delta_0^2} (3u_0 + 23t_1^2 + 220t_2) \right. \\
& + \frac{3}{256\Delta_0^3} (-1468t_1^3 v_0 u_0 + 1064t_1^2 t_2 u_0^2 + 27369t_1^2 v_0^2 - 45828t_1 t_2^2 v_0 - 25764t_1 t_2 v_0 u_0 \\
& - 1504t_1 v_0 u_0^2 + 3080t_2^3 u_0 + 3692t_2^2 u_0^2 + 48105t_2 v_0^2 + 1643v_0^2 u_0) \\
& + \frac{2187v_0^2}{256\Delta_0^4} (-478t_1^3 v_0 u_0 + 692t_1^2 t_2 u_0^2 + 3123t_1^2 v_0^2 - 5922t_1 t_2^2 v_0 - 3624t_1 t_2 v_0 u_0 \\
& - 382t_1 v_0 u_0^2 + 884t_2^3 u_0 + 944t_2^2 u_0^2 + 3690t_2 v_0^2 + 257v_0^2 u_0) \\
& + \frac{14348907v_0^4}{256\Delta_0^5} (-2t_1^3 v_0 u_0 + 4t_1^2 t_2 u_0^2 + 9t_1^2 v_0^2 - 18t_1 t_2^2 v_0 - 12t_1 t_2 v_0 u_0 - 2t_1 v_0 u_0^2 \\
& + 4t_2^3 u_0 + 4t_2^2 u_0^2 + 9t_2 v_0^2 + v_0^2 u_0) \}
\end{aligned}$$

Instanton corrections for $N_f = 3$ are ($t_1 = m_1 + m_2 + m_3$, $t_2 = m_1 m_2 + m_1 m_3 + m_2 m_3$, $t_3 = m_1 m_2 m_3$):

$$\begin{aligned}
\mathcal{F}_1^{N_f=3} &= \frac{1}{2\pi i} \frac{1}{4\Delta_0} (-3u_0 v_0 + 2u_0^2 t_1 - 9v_0 t_2 + 6u_0 t_3) \\
\mathcal{F}_2^{N_f=3} &= \frac{1}{4\pi i} \left\{ \frac{1}{128\Delta_0} (5t_1^2 + 6t_2 - 3u_0) \right. \\
& + \frac{9}{64\Delta_0^2} (51t_1^2 v_0^2 - 46t_1 t_2 v_0 u_0 + 28t_1 t_3 u_0^2 - 6t_1 v_0 u_0^2 + 2t_2^2 u_0^2 - 222t_2 t_3 v_0 + 156t_2 v_0^2 \\
& + 34t_3^2 u_0 - 118t_3 v_0 u_0 + 10v_0^2 u_0) \\
& + \frac{3645v_0^2}{128\Delta_0^3} (9t_1^2 v_0^2 - 12t_1 t_2 v_0 u_0 + 8t_1 t_3 u_0^2 - 4t_1 v_0 u_0^2 + 4t_2^2 u_0^2 - 36t_2 t_3 v_0 + 18t_2 v_0^2 \\
& + 12t_3^2 u_0 - 12t_3 v_0 u_0 + 3v_0^2 u_0) \} \\
\mathcal{F}_3^{N_f=3} &= \frac{1}{6\pi i} \left\{ \frac{1}{512\Delta_0^2} (138t_1 t_2^2 + 1320t_1^2 t_3 + 1512t_2 t_3 + 18t_1^3 u_0 + 60t_1 t_2 u_0 + 108t_3 u_0 \right.
\end{aligned}$$

$$\begin{aligned}
& -14t_1u_0^2 - 201t_1^2v_0 - 252t_2v_0 + 21u_0v_0) \\
& + \frac{3}{512\Delta_0^3} (6160t_3^3u_0 + 2128t_2^2t_3u_0^2 + 7384t_1t_2^2u_0^2 - 91656t_2t_3^2v_0 - 2936t_2^3u_0v_0 \\
& - 51528t_1t_2t_3u_0v_0 - 49956t_3^2u_0v_0 - 3008t_1^2t_2u_0^2v_0 - 3044t_2^2u_0^2v_0 - 19120t_1t_3u_0^2v_0 \\
& + 54738t_1t_2^2v_0^2 + 96210t_1^2t_3v_0^2 + 231624t_2t_3v_0^2 + 3286t_1^3u_0v_0^2 + 28896t_1t_2u_0v_0^2 \\
& + 35778t_3u_0v_0^2 + 1126t_1u_0^2v_0^2 - 33039t_1^2v_0^3 - 47943t_2v_0^3 - 1067u_0v_0^3) \\
& + \frac{2187v_0^2}{512\Delta_0^4} (1768t_3^3u_0 + 1384t_2^2t_3u_0^2 + 1888t_1t_2^2u_0^2 - 11844t_2t_3^2v_0 - 956t_2^3u_0v_0 \\
& - 7248t_1t_2t_3u_0v_0 - 5136t_3^2u_0v_0 - 764t_1^2t_2u_0^2v_0 - 1016t_2^2u_0^2v_0 - 2536t_1t_3u_0^2v_0 \\
& + 6246t_1t_2^2v_0^2 + 7380t_1^2t_3v_0^2 + 17028t_2t_3v_0^2 + 514t_1^3u_0v_0^2 + 3840t_1t_2u_0v_0^2 \\
& + 2676t_3u_0v_0^2 + 418t_1u_0^2v_0^2 - 3285t_1^2v_0^3 - 3852t_2v_0^3 - 275u_0v_0^3) \\
& + \frac{14348907v_0^4}{512\Delta_0^5} (8t_3^3u_0 + 8t_2^2t_3u_0^2 + 8t_1t_2^2u_0^2 - 36t_2t_3^2v_0 - 4t_2^3u_0v_0 - 24t_1t_2t_3u_0v_0 \\
& - 12t_3^2u_0v_0 - 4t_1^2t_2u_0^2v_0 - 4t_2^2u_0^2v_0 - 8t_1t_3u_0^2v_0 + 18t_1t_2^2v_0^2 + 18t_1^2t_3v_0^2 \\
& + 36t_2t_3v_0^2 + 2t_1^3u_0v_0^2 + 12t_1t_2u_0v_0^2 + 6t_3u_0v_0^2 + 2t_1u_0^2v_0^2 - 9t_1^2v_0^3 - 9t_2v_0^3 - u_0v_0^3) \}
\end{aligned}$$

Starting with $N_f = 4$ we calculated the instanton corrections for equal masses only. Nevertheless for the one instanton contributions we can infer the general result because in this case the dependence on t_i is linear (for equal masses: $t_1 = 4m$, $t_2 = 6m^2$, $t_3 = 4m^3$, $t_4 = m^4$):

$$\begin{aligned}
\mathcal{F}_1^{N_f=4} &= \frac{1}{2\pi i} \frac{1}{4\Delta_0} (-2u_0^3 + 18v_0^2 - 3u_0v_0t_1 + 2u_0^2t_2 - 9v_0t_3 + 6u_0t_4) \\
\mathcal{F}_2^{N_f=4} &= \frac{1}{4\pi i} \left\{ \frac{1}{128\Delta_0} (u_0^2 - 12u_0m^2 - 12v_0m + 366m^4) \right. \\
&+ \frac{9}{64\Delta_0^2} (u_0^2v_0^2 - 264u_0^2v_0m^3 + 200u_0^2m^6 + 388u_0v_0^2m^2 - 1576u_0v_0m^5 + 34u_0m^8 \\
&- 240v_0^3m + 4650v_0^2m^4 - 888v_0m^7) \\
&+ \frac{3645v_0^2}{128\Delta_0^3} (u_0^2v_0^2 - 112u_0^2v_0m^3 + 112u_0^2m^6 + 84u_0v_0^2m^2 - 336u_0v_0m^5 + 12u_0m^8 \\
&- 36v_0^3m + 630v_0^2m^4 - 144v_0m^7) \} \\
\mathcal{F}_3^{N_f=4} &= \frac{1}{6\pi i} \left\{ \frac{15m^2}{512\Delta_0} + \frac{3}{1024\Delta_0^2} (59280m^8 + 14800m^6u_0 + 360m^4u_0^2 - 87792m^5v_0 \right. \\
&- 2480m^3u_0v_0 - 40mu_0^2v_0 + 4092m^2v_0^2 + 15u_0v_0^2) \\
&+ \frac{3}{1024\Delta_0^3} (12320m^{12}u_0 + 156704m^{10}u_0^2 - 733248m^{11}v_0 - 3248800m^9u_0v_0 \\
&- 2431488m^7u_0^2v_0 + 25314660m^8v_0^2 + 10245648m^6u_0v_0^2 + 908340m^4u_0^2v_0^2 \\
&- 24334128m^5v_0^3 - 1240600m^3u_0v_0^3 - 10552mu_0^2v_0^3 + 961920m^2v_0^4 + 2651u_0v_0^4) \\
&+ \frac{2187v_0^2}{1024\Delta_0^4} (3536m^{12}u_0 + 66944m^{10}u_0^2 - 94752m^{11}v_0 - 511360m^9u_0v_0
\end{aligned}$$

$$\begin{aligned}
& -500160m^7u_0^2v_0 + 2299320m^8v_0^2 + 1203888m^6u_0v_0^2 + 164400m^4u_0^2v_0^2 \\
& -1952208m^5v_0^3 - 143680m^3u_0v_0^3 - 1816mu_0^2v_0^3 + 79236m^2v_0^4 + 293u_0v_0^4) \\
& + \frac{14348907v_0^4}{1024\Delta_0^5} \left(16m^{12}u_0 + 352m^{10}u_0^2 - 288m^{11}v_0 - 1760m^9u_0v_0 - 2112m^7u_0^2v_0 \right. \\
& + 5940m^8v_0^2 + 3696m^6u_0v_0^2 + 660m^4u_0^2v_0^2 - 4752m^5v_0^3 - 440m^3u_0v_0^3 - 8mu_0^2v_0^3 \\
& \left. + 198m^2v_0^4 + u_0v_0^4 \right) \}
\end{aligned}$$

Finally for $N_f = 5$ the instanton corrections are:

$$\begin{aligned}
\mathcal{F}_1^{N_f=5} &= \frac{1}{2\pi i} \frac{1}{4\Delta_0} (6u_0t_5 - 9v_0t_4 + 2u_0^2t_3 - 3u_0v_0t_2 - 2u_0^3t_1 + 18v_0^2t_1 - v_0u_0^2) \\
\mathcal{F}_2^{N_f=5} &= \frac{1}{4\pi i} \left\{ -\frac{15}{512} \right. \\
&+ \frac{1}{512\Delta_0} (5000m^6 + 520m^4u_0 + 660m^2u_0^2 - 3440m^3v_0 - 620mu_0v_0 + 227v_0^2) \\
&+ \frac{9}{512\Delta_0^2} (272m^{10}u_0 + 2640m^8u_0^2 - 8880m^9v_0 - 27840m^7u_0v_0 - 11424m^5u_0^2v_0 \\
&+ 115920m^6v_0^2 + 26880m^4u_0v_0^2 + 840m^2u_0^2v_0^2 - 33120m^3v_0^3 - 520mu_0v_0^3 + 111v_0^4) \\
&+ \frac{10935v_0^2}{512\Delta_0^3} (16m^{10}u_0 + 240m^8u_0^2 - 240m^9v_0 - 960m^7u_0v_0 - 672m^5u_0^2v_0 + 2520m^6v_0^2 \\
&+ 840m^4u_0v_0^2 + 60m^2u_0^2v_0^2 - 720m^3v_0^3 - 20mu_0v_0^3 + 3v_0^4) \} \\
\mathcal{F}_3^{N_f=5} &= \frac{1}{6\pi i} \left\{ \frac{1}{3072\Delta_0} (16944m^5 + 630m^3u_0 - 3585m^2v_0 + 340mu_0^2 - 150v_0u_0) \right. \\
&+ \frac{3}{2048\Delta_0^2} (349200m^{11} + 208800m^9u_0 - 1973520m^8v_0 + 49840m^7u_0^2 \\
&- 399320m^6v_0u_0 + 1270782m^5v_0^2 - 62000m^4v_0u_0^2 + 114830m^3v_0^2u_0 \\
&- 104715m^2v_0^3 + 1590mv_0^2u_0^2 - 497v_0^3u_0) \\
&+ \frac{3}{2048\Delta_0^3} (24640m^{15}u_0 - 1833120m^{14}v_0 + 508160m^{13}u_0^2 - 13771840m^{12}v_0u_0 \\
&+ 144205560m^{11}v_0^2 - 20073184m^{10}v_0u_0^2 + 122906920m^9v_0^2u_0 - 444593340m^8v_0^3 \\
&+ 33250440m^7v_0^2u_0^2 - 80967620m^6v_0^3u_0 + 129109590m^5v_0^4 - 7523380m^4v_0^3u_0^2 \\
&+ 7198430m^3v_0^4u_0 - 4319415m^2v_0^5 + 155750mv_0^4u_0^2 - 25835v_0^5u_0) \\
&+ \frac{2187v_0^2}{2048\Delta_0^4} (7072m^{15}u_0 - 236880m^{14}v_0 + 213920m^{13}u_0^2 - 2133040m^{12}v_0u_0 \\
&+ 12841920m^{11}v_0^2 - 3923920m^{10}v_0u_0^2 + 13533520m^9v_0^2u_0 - 33050160m^8v_0^3 \\
&+ 4667520m^7v_0^2u_0^2 - 7127120m^6v_0^3u_0 + 7873866m^5v_0^4 - 495040m^4v_0^3u_0^2 \\
&+ 315770m^3v_0^4u_0 - 131985m^2v_0^5 + 2450mv_0^4u_0^2 - 311v_0^5u_0) \\
&+ \frac{14348907v_0^4}{2048\Delta_0^5} (32m^{15}u_0 - 720m^{14}v_0 + 1120m^{13}u_0^2 - 7280m^{12}v_0u_0 + 32760m^{11}v_0^2
\end{aligned}$$

$$\begin{aligned} & -16016m^{10}v_0u_0^2 + 40040m^9v_0^2u_0 - 77220m^8v_0^3 + 17160m^7v_0^2u_0^2 - 20020m^6v_0^3u_0 \\ & + 18018m^5v_0^4 - 1820m^4v_0^3u_0^2 + 910m^3v_0^4u_0 - 315m^2v_0^5 + 10mv_0^4u_0^2 - v_0^5u_0 \bigg\} \end{aligned}$$

The results for different N_f are related by the decoupling limit where one takes a single hypermultiplet mass to infinity keeping $m\Lambda_{N_f}^{2N_c-N_f} \equiv \Lambda_{N_f-1}^{2N_c-N_f+1}$ fixed. This amounts for the expressions above in taking from $\mathcal{F}_n^{N_f}$ only the terms of order n in t_i followed by the transformations $t_1 \rightarrow 1$, $t_{i+1} \rightarrow t_i$ and $\Lambda^{2N_c-N_f} \rightarrow \Lambda^{2N_c-N_f+1}$.

Taking the massless limit $m_i \rightarrow 0$ (or $m \rightarrow -\frac{\Lambda}{12}$ in the case $N_f = 5$) we reproduce our previous results [20].

We checked that for $N_f = 1, 2, 3$ the one and two instanton corrections agree with the formulas given in [2]. For $N_f = 4, 5$ we find coincidence after performing the shift mentioned before. Our results coincide with the massless one instanton calculation done in [19].

4 Generalization to other Gauge Groups

This method of calculating instanton corrections of arbitrary order from PF-operators can be generalized to other gauge groups like $SO(2r+1)$, $SO(2r)$ and $Sp(2r)$, starting e.g. from the curves given in [7, 10, 12].

To fix the linear combination of the power series solutions ω_i to the PF-operators we look at the classical limit $\Lambda \rightarrow 0$ of the curves which in all cases mentioned above contains a factor $\prod_{i=1}^r (x^2 - a_i^2)$, where r is the rank of the gauge group G . Performing a Miura transformation we get:

$$\prod_{i=1}^r (x^2 - a_i^2) = x^{2r} - \sum_{i=1}^r u_{2i} x^{2(r-i)} \quad (25)$$

where u_{2i} are the gauge invariant Casimirs of order $2i$, only in the $SO(2r)$ case $u_{2r} = t^2$ where t is the exceptional Casimir of order r . Solving this equation in the limit $u_2 \rightarrow \infty$ we obtain the leading terms of the periods $a_i(u)$. Comparing these expressions for $a_i(u)$ with the ω_i fixes the linear combination completely.

To derive the formula corresponding to (23) for $u_2(a)$ with the above gauge groups G we use the perturbative part of the prepotential given in [14] and the expression for the instanton corrections as a series in Λ . Inserting the corresponding roots and weights of the gauge group G into (1), the one loop part of \mathcal{F}^G is:

$$\begin{aligned} \mathcal{F}_{1-\text{loop}}^G &= \frac{i}{4\pi} \left\{ \sum_{k < l}^r \sum_{\epsilon = \pm} (a_k + \epsilon a_l)^2 \ln \frac{(a_k + \epsilon a_l)^2}{\Lambda^2} + \xi \sum_{k=1}^r a_k^2 \ln \frac{a_k^2}{\Lambda^2} \right. \\ &\quad \left. - \frac{1}{2} \sum_{k=1}^r \sum_{j=1}^{N_f} \sum_{\epsilon = \pm} (\epsilon a_k + m_j)^2 \ln \frac{(\epsilon a_k + m_j)^2}{\Lambda^2} \right\} \end{aligned} \quad (26)$$

where ξ takes the value 1 for $SO(2r+1)$, 0 for $SO(2r)$ and 4 for $Sp(2r)$. The instanton part is

$$\mathcal{F}_{\text{inst}}^G = \sum_n \mathcal{F}_n \Lambda^{(2r-2+\xi-N_f)2n} \quad (27)$$

The exponent of Λ in this series is associated to the beta function. After taking the derivative of \mathcal{F}^G with respect to Λ we find:

$$\begin{aligned} u_2 &= \frac{2\pi i}{(2r-2+\xi-N_f)} \left(\Lambda \frac{\partial \mathcal{F}^G}{\partial \Lambda} + \frac{r}{2\pi i} \sum_{j=1}^{N_f} m_j^2 \right) \\ &= u_0 + 2\pi i \sum_{n=1} \mathcal{F}_n 2n \Lambda^{2n(2r-2+\xi-N_f)} \end{aligned} \quad (28)$$

where $u_0 = \sum_{k=1}^r a_k^2$. The instanton corrections to the prepotential \mathcal{F}^G can then be obtained by performing the same procedure as for the gauge group $SU(3)$.

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Appendix: PF-Operators for $N_f = 1$

In this appendix we give as an example the complete set of PF-operators $\mathcal{L}_{(i)} \int \partial_v \lambda = 0$ and $\hat{\mathcal{L}}_{(i)} \int \partial_u \lambda = 0$ for the case of $N_f = 1$ massive hypermultiplet:

$$\begin{aligned} \mathcal{L}_{(1)} &= \left(c_1^{(1)} \partial_{uu}^2 + c_2^{(1)} \partial_{uv}^2 + c_4^{(1)} \partial_u + c_5^{(1)} \partial_v + c_6^{(1)} \right) \\ \mathcal{L}_{(2)} &= \left(c_3^{(2)} \partial_{vv}^2 + c_2^{(2)} \partial_{uv}^2 + c_4^{(2)} \partial_u + c_5^{(2)} \partial_v + c_6^{(2)} \right) \\ \hat{\mathcal{L}}_{(1)} &= \left(\hat{c}_1^{(1)} \partial_{uu}^2 + \hat{c}_2^{(1)} \partial_{uv}^2 + \hat{c}_4^{(1)} \partial_u + \hat{c}_5^{(1)} \partial_v + \hat{c}_6^{(1)} \right) \\ \hat{\mathcal{L}}_{(2)} &= \left(\hat{c}_3^{(2)} \partial_{vv}^2 + \hat{c}_2^{(2)} \partial_{uv}^2 + \hat{c}_4^{(2)} \partial_u + \hat{c}_5^{(2)} \partial_v + \hat{c}_6^{(2)} \right) \end{aligned} \quad (29)$$

For $\mathcal{L}_{(1)}$ we get:

$$\begin{aligned} c_1^{(1)} &= 4 \left(9375 \Lambda^{10} m_1^2 + 2 \Lambda^5 (-550 u^3 m_1 + 750 u^2 v + 15420 u^2 m_1^3 - 31050 u v m_1^2 - 35802 u m_1^5 \right. \\ &\quad + 6750 v^2 m_1 + 37665 v m_1^4 + 17496 m_1^7) + 4 (-1036 u^5 m_1^2 + 2088 u^4 v m_1 + 2472 u^4 m_1^4 \\ &\quad - 1260 u^3 v^2 - 3276 u^3 v m_1^3 - 1296 u^3 m_1^6 - 8235 u^2 v^2 m_1^2 + 1296 u^2 v m_1^5 + 17010 u v^3 m_1 \\ &\quad \left. + 17334 u v^2 m_1^4 - 6075 v^4 - 18711 v^3 m_1^3 - 8748 v^2 m_1^6) \right) \\ c_2^{(1)} &= 625 \Lambda^{10} (11 u m_1 - 15 v + 54 m_1^3) + 4 \Lambda^5 (6815 u^3 m_1^2 - 18450 u^2 v m_1 - 23346 u^2 m_1^4 \\ &\quad + 12375 u v^2 + 43290 u v m_1^3 + 9720 u m_1^6 - 35775 v^2 m_1^2 - 7776 v m_1^5) + 16 (68 u^5 m_1^3 \\ &\quad - 3228 u^4 v m_1^2 + 72 u^4 m_1^5 + 6192 u^3 v^2 m_1 + 7416 u^3 v m_1^4 - 3240 u^2 v^3 - 9531 u^2 v^2 m_1^3 \end{aligned}$$

$$\begin{aligned}
& -3888u^2vm_1^6 - 1053uv^3m_1^2 + 1458uv^2m_1^5 + 4860v^4m_1 + 972v^3m_1^4) \\
c_4^{(1)} &= 16(5\Lambda^5(-65u^2m_1 + 225uv + 438um_1^3 - 1170vm_1^2 - 324m_1^5) + 2(-1558u^4m_1^2 \\
& + 3174u^3vm_1 + 3738u^3m_1^4 - 2160u^2v^2 - 5697u^2vm_1^3 - 1944u^2m_1^6 + 189uv^2m_1^2 \\
& + 1296uvm_1^5 + 3240v^3m_1 + 648v^2m_1^4)) \\
c_5^{(1)} &= 4(-3125\Lambda^{10}m_1 + 90\Lambda^5(10u^2m_1^2 - 95uvm_1 - 164um_1^4 + 125v^2 + 189vm_1^3 + 108m_1^6) \\
& + 4(170u^4m_1^3 - 5751u^3vm_1^2 + 180u^3m_1^5 + 10908u^2v^2m_1 + 13068u^2vm_1^4 - 6075uv^3 \\
& - 16740uv^2m_1^3 - 6804uvm_1^6 + 2673v^3m_1^2 + 972v^2m_1^5)) \\
c_6^{(1)} &= 48(25\Lambda^5(3um_1 + 5v - 4m_1^3) + 2(-174u^3m_1^2 + 362u^2vm_1 + 422u^2m_1^4 - 300uv^2 \\
& - 705uvm_1^3 - 216um_1^6 + 297v^2m_1^2 + 108vm_1^5))
\end{aligned}$$

The second PF-operator $\mathcal{L}_{(2)}$ is:

$$\begin{aligned}
c_2^{(2)} &= 28125\Lambda^{10}m_1 + 108\Lambda^5(1025u^2m_1^2 - 2250uvm_1 - 1910um_1^4 + 625v^2 + 2200vm_1^3 \\
& + 648m_1^6) + 16(32u^5m_1 - 240u^4v - 540u^4m_1^3 - 9792u^3vm_1^2 + 648u^3m_1^5 + 20520u^2v^2m_1 \\
& + 22680u^2vm_1^4 - 8100uv^3 - 23571uv^2m_1^3 - 11664uvm_1^6 - 2430v^3m_1^2 + 1458v^2m_1^5) \\
c_3^{(2)} &= c_1^{(1)} \\
c_4^{(2)} &= 48(75\Lambda^5m_1(-15u - 2m_1^2) + 2(40u^4 - 1678u^3m_1^2 + 3330u^2vm_1 + 3798u^2m_1^4 - 1350uv^2 \\
& - 4023uvm_1^3 - 1944um_1^6 - 540v^2m_1^2 + 324vm_1^5)) \\
c_5^{(2)} &= 8(15\Lambda^5(-50u^2 + 20um_1^2 - 825vm_1 + 198m_1^4) + 2(80u^4m_1 - 240u^3v - 1350u^3m_1^3 \\
& - 18603u^2vm_1^2 + 1620u^2m_1^5 + 39960uv^2m_1 + 40824uvm_1^4 - 14175v^3 - 45198v^2m_1^3 \\
& - 20412vm_1^6)) \\
c_6^{(2)} &= 48(-375\Lambda^5m_1 + 2(40u^3 - 642u^2m_1^2 + 1290uvm_1 + 1326um_1^4 - 450v^2 - 1557vm_1^3 \\
& - 648m_1^6))
\end{aligned}$$

The PF-operators $\hat{\mathcal{L}}_{(i)}$ for the differential $\partial_u \lambda$ are:

$$\begin{aligned}
\hat{c}_1^{(1)} &= 40625\Lambda^{15}m_1 + 40\Lambda^{10}(-100u^3 - 75u^2m_1^2 - 3075uvm_1 + 6102um_1^4 - 1125v^2 - 2835vm_1^3 \\
& - 5832m_1^6) + 16\Lambda^5(-2320u^5m_1 + 2000u^4v + 22941u^4m_1^3 - 28050u^3vm_1^2 - 45900u^3m_1^5 \\
& + 12555u^2v^2m_1 + 24246u^2vm_1^4 + 23328u^2m_1^7 + 9450uv^3 - 17955uv^2m_1^3 + 17496uvm_1^6 \\
& + 7695v^3m_1^2 + 14580v^2m_1^5) + 64u(-680u^6m_1^2 + 1680u^5vm_1 + 1588u^5m_1^4 - 840u^4v^2 \\
& - 2760u^4vm_1^3 - 864u^4m_1^6 - 4230u^3v^2m_1^2 + 864u^3vm_1^5 + 7452u^2v^3m_1 + 10017u^2v^2m_1^4 \\
& - 4050uv^4 - 4212uv^3m_1^3 - 5832uv^2m_1^6 - 243v^4m_1^2 - 4374v^3m_1^5) \\
\hat{c}_2^{(1)} &= 2(625\Lambda^{15}(5u + 39m_1^2) + 2\Lambda^{10}(14600u^3m_1 - 15500u^2v - 63420u^2m_1^3 + 30000uvm_1^2 \\
& + 119880um_1^5 - 30375v^2m_1 - 63180vm_1^4 - 69984m_1^7) + 8\Lambda^5(7090u^5m_1^2 - 19020u^4vm_1 \\
& - 18690u^4m_1^4 + 11250u^3v^2 + 37380u^3vm_1^3 + 9720u^3m_1^6 + 540u^2v^2m_1^2 - 3564u^2vm_1^5
\end{aligned}$$

$$\begin{aligned}
& -2430uv^3m_1 - 64638uv^2m_1^4 + 4050v^4 + 33615v^3m_1^3 + 34992v^2m_1^6) + 32(20u^7m_1^3 \\
& - 2080u^6vm_1^2 + 24u^6m_1^5 + 4380u^5v^2m_1 + 4620u^5vm_1^4 - 2160u^4v^3 - 6471u^4v^2m_1^3 \\
& - 2592u^4vm_1^6 + 1404u^3v^3m_1^2 + 810u^3v^2m_1^5 - 1377u^2v^4m_1 - 81u^2v^3m_1^4 + 8505uv^4m_1^3 \\
& - 4374v^5m_1^2 - 4374v^4m_1^5)) \\
\hat{c}_4^{(1)} &= 4(25\Lambda^{10}(-80u^2 + 1146um_1^2 - 585vm_1) + 8\Lambda^5(-2320u^4m_1 + 2000u^3v - 6036u^3m_1^3 \\
& + 18675u^2vm_1^2 + 19440u^2m_1^5 - 12150uv^2m_1 - 35451uvm_1^4 - 11664um_1^7 + 2025v^3 \\
& + 13635v^2m_1^3 + 5832vm_1^6) + 16(-1360u^6m_1^2 + 3360u^5vm_1 + 3318u^5m_1^4 - 1680u^4v^2 \\
& - 6141u^4vm_1^3 - 1728u^4m_1^6 + 7308u^3v^2m_1^2 + 1350u^3vm_1^5 - 10287u^2v^3m_1 - 10692u^2v^2m_1^4 \\
& + 4050uv^4 + 18549uv^3m_1^3 + 5832uv^2m_1^6 - 7047v^4m_1^2 - 2916v^3m_1^5)) \\
\hat{c}_5^{(1)} &= 8(5\Lambda^{10}(20u^2m_1 - 375uv - 744um_1^3 + 390vm_1^2) + 4\Lambda^5(2840u^4m_1^2 - 5910u^3vm_1 \\
& - 8400u^3m_1^4 + 3750u^2v^2 + 8574u^2vm_1^3 + 5184u^2m_1^6 - 2610uv^2m_1^2 + 1620uvm_1^5 \\
& + 135v^3m_1 - 891v^2m_1^4) + 8(40u^6m_1^3 - 2630u^5vm_1^2 + 48u^5m_1^5 + 4980u^4v^2m_1 \\
& + 5880u^4vm_1^4 - 2430u^3v^3 - 7056u^3v^2m_1^3 - 3240u^3vm_1^6 + 3780u^2v^3m_1^2 - 162u^2v^2m_1^5 \\
& - 810uv^4m_1 - 810uv^3m_1^4 + 243v^4m_1^3)) \\
\hat{c}_6^{(1)} &= 8(-125\Lambda^{10}u + 4\Lambda^5(-290u^3m_1 + 250u^2v - 474u^2m_1^3 + 150uvm_1^2 + 540um_1^5 \\
& + 45v^2m_1 - 189vm_1^4) + 8(-170u^5m_1^2 + 420u^4vm_1 + 468u^4m_1^4 - 210u^3v^2 - 888u^3vm_1^3 \\
& - 216u^3m_1^6 + 864u^2v^2m_1^2 + 162u^2vm_1^5 - 270uv^3m_1 - 270uv^2m_1^4 + 81v^3m_1^3))
\end{aligned}$$

For $\hat{\mathcal{L}}_{(2)}$ we get:

$$\begin{aligned}
\hat{c}_2^{(2)} &= 2(15625\Lambda^{15} + 10\Lambda^{10}(1300u^2m_1 - 8250uv + 9630um_1^3 - 2925vm_1^2 - 11664m_1^5) \\
& + 8\Lambda^5(9530u^4m_1^2 - 18780u^3vm_1 - 17046u^3m_1^4 + 22050u^2v^2 - 17505u^2vm_1^3 \\
& + 5832u^2m_1^6 + 7560uv^2m_1^2 + 42768uvm_1^5 - 2025v^3m_1 + 4860v^2m_1^4) \\
& + 32u(320u^6m_1 - 160u^5v - 924u^5m_1^3 - 6024u^4vm_1^2 + 648u^4m_1^5 + 10188u^3v^2m_1 \\
& + 14256u^3vm_1^4 - 5400u^2v^3 - 8991u^2v^2m_1^3 - 7776u^2vm_1^6 + 810uv^3m_1^2 - 3402uv^2m_1^5 \\
& + 1215v^4m_1 - 2916v^3m_1^4)) \\
\hat{c}_3^{(2)} &= \hat{c}_1^{(1)} \\
\hat{c}_4^{(2)} &= 32(125\Lambda^{10}(-10u - 3m_1^2) + 15\Lambda^5(120u^3m_1 + 220u^2v - 572u^2m_1^3 + 465uvm_1^2 + 162um_1^5 \\
& - 180v^2m_1 + 432vm_1^4) + 2u(-6704u^4m_1^2 + 10752u^3vm_1 + 15102u^3m_1^4 - 5400u^2v^2 \\
& - 11745u^2vm_1^3 - 7776u^2m_1^6 + 1512uv^2m_1^2 - 1782uvm_1^5 + 2025v^3m_1 - 4860v^2m_1^4)) \\
\hat{c}_5^{(2)} &= 8(25\Lambda^{10}(20um_1 - 375v + 36m_1^3) + 10\Lambda^5(-176u^3m_1^2 - 192u^2vm_1 + 504u^2m_1^4 + 3150uv^2 \\
& - 6210uvm_1^3 + 2673v^2m_1^2 + 4860vm_1^5) + 16u(320u^5m_1 - 160u^4v - 924u^4m_1^3 - 4167u^3vm_1^2 \\
& + 648u^3m_1^5 + 6102u^2v^2m_1 + 9504u^2vm_1^4 - 3375uv^3 - 3888uv^2m_1^3 - 4860uvm_1^6 - 243v^3m_1^2 \\
& - 3645v^2m_1^5))
\end{aligned}$$

$$\hat{c}_6^{(2)} = 8 \left(-625\Lambda^{10} + 30\Lambda^5(70uv - 134um_1^3 + 69vm_1^2 + 108m_1^5) + 16u(-401u^3m_1^2 + 438u^2vm_1 + 798u^2m_1^4 - 225uv^2 - 360uvm_1^3 - 324um_1^6 - 27v^2m_1^2 - 243vm_1^5) \right)$$

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